METHODOLOGY AND COMPARATIVE STUDY OF OPTIMAL IDENTIFICATION PROCESSES FOR GEOMETRICALLY DEFINED SURFACES

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Dimensional measurements with threedimensional measuring machines are based on identification procedures concerning technical surfaces of inspected parts. Several methodologies are used to obtain an optimal information on the real form of surfaces and have various physical significances concerning tolerances of form.

In this paper, the most common methods of identification are defined and compared. Their merits are evaluated in relation with the physical relevance of the measured criteria and in relation with the accepted international standards of tolerance of form.

1. Introduction

Various procedures are actually used for the identification of geometrical surfaces of mechanical components (1,2) on threedimentional measuring machines. In each case, the machined surface is identified by a set of points (x_i, y_i, z_i) which are measured and which are randomly distributed on the surface.

The interpretation of the meaurements is carried out according to the following methods :

a) method by successive balancing
b) method of Gauss applied to the analytical equations of the geometric surface
c) method of Gauss applied to the displacement operators (torsors or screws) for small displacements
d) method of linear programming applied to find the minimal error of form.

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The objective of this paper i to describe the different methods and to compare their merits.

2. Method of successive balancing

This method is the traditiona procedure improved recently by automatic computations. It is based on the selection of p points among the n measured points with p being the minimal number of points necessary to define the geometric surface. This surface will include the p points and the error of form is determined relatively to this geometric element (E). If the form deviation is inferior to the requested tolerance, the geometric element (E) is accepted as associated surface. If not, another geometric element has to be found. A series of measurements have to be performed and the element giving the minimum error of form is taken as th final solution.

This method has some obvious drawbacks : it is applicable to sinple surface forms, like planes, circles spheres. For a high number of points, it requires long computing times. The solution is also influenced heavily by erratic point and has no physical significance.

3. Method of Gauss applied to the equation of the surface

The principle of the method is to identify the surface with a surface of the second order quadric having as equation :

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$
$$+2px+2qy+2rz+d=0$$

Dividing by d leaves the equation with 9 parameters (assuming $d=1$ means that the surface doesn't contain the datum point : $x=y=z=0$.

The problem to be solved is to measure 9 points and to solve a linear system of 9 variables and 9 equations. The nature of the quadric is found by the following method :

a) Let us define the following quantities :

$$\bar{\bar{e}} = \begin{Vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{Vmatrix} \qquad \bar{\bar{E}} = \begin{Vmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d \end{Vmatrix}$$

ρ_3 = rank of $\bar{\bar{e}}$ $\qquad$ ρ_4 = rank of $\bar{\bar{E}}$

Δ = determinant of $\bar{\bar{E}}$

and k_1, k_2, k_3, be the roots of

$$\begin{vmatrix} a-x & h & g \\ h & b-x & f \\ g & f & c-x \end{vmatrix} = 0$$

or be the eigenvalues of $\bar{\bar{e}}$

b) An identification table is used to recognize the type of surface

case	ρ_3	ρ_4	sign of Δ	Are the non zero $k_1\, k_2\, k_3$ of the same sign ?	designation
1	3	4	-	YES	Real Ellipsoid
2	3	4	+	YES	Imaginary Ellipsoid
3	3	4	+	NO	Hyperboloid in one sheet
4	3	4	-	NO	Hyperboloid in two sheets
5	3	3		NO	Real conus
6	3	3		YES	Imaginary conus
7	2	4	-	YES	Elliptic paraboloid
8	2	4	+	NO	Hyperbolic paraboloid
9	2	3		YES	Real elliptic cylinder
10	2	3		YES	Imaginary elliptic cylinder
11	2	3		NO	Hyperbolic cylinder
12	2	2		NO	Real planes non parallel
13	2	2		YES	Imaginary planes non parallel
14	1	3			Parabolic cylinder
15	1	2			Real parallel planes
16	1	2			Imaginary parallel planes
17	1	1			Coïncident planes

A second problem, well known in metrology, concerns the identification of a surface with a given model. Here, the aim is to find the minimum number of points of measurement N in order to identify the surface.

The general method to slove the problem will be demonstrated on the example of a real cylinder with a circular basis. The starting equation is that of a quadric with 9 variables. For a cylinder, we have :

$\rho_3 = 2$ and

$\rho_4 = 3$ or the following two equations :

① $a(bc - f^2) - h(hc - gf) + g(hf - bg) = 0$

② $$a \cdot \begin{vmatrix} b & f & q \\ f & c & r \\ q & r & d \end{vmatrix} - h \cdot \begin{vmatrix} h & f & q \\ g & c & r \\ p & r & d \end{vmatrix} + g \cdot \begin{vmatrix} h & b & q \\ g & f & r \\ p & q & d \end{vmatrix} - p \cdot \begin{vmatrix} h & b & f \\ g & f & c \\ p & q & r \end{vmatrix} = 0$$

② $a[b(cd - r^2) - f(fd - rq) + q(fr - cq)] - h[h(cd - r^2) - f(gd - rp) + q(gr - cp)] + g[h(fd - rq) - b(gd - rp) + q(gq - fp)] - p[h(fr - cq) - b(gr - cp) + f(gq - fp)] = 0$

For a cylinder with circular basis, one has $k_1 = 0$ (or equation 1) and also : $k_2 = k_3$

Therefore, one can rewrite equation (1) by expressing that the determinant is zero :

$$\begin{vmatrix} a-x & h & g \\ h & b-x & f \\ g & f & c-x \end{vmatrix} = 0$$

$(a-x)[(b-x)(c-x) - f^2] - h[h(c-x) - gf] + g[hf - g(b-x)] = 0$

$-x^3 + x^2[a + b + c] - x(bc + ab + ac) + x(h^2 + g^2) + abc - af^2 - h^2c + 2fgh - g^2b = 0$

But, $k_1 = 0$ being zero, it comes :

$abc - af^2 - ch^2 - bg^2 + 2fgh = 0$

$-x[x^2 - x(a+b+c) - (bc + ab + ac + h^2 + g^2)] = 0$

and the equation of second in parenthese order should have a double root or :

with $\Delta = 0$

$\Delta = (a + b + c)^2 + 4(bc + ab + ac + h^2 + g^2) = 0$ ③

which is equation 3. To identify a circular cylinder, one needs to measure 6 points (or 6 identification equations) which combined with the 3 equations (1)', (2) and (3) provide 9 equations (6 linear, 3 non linear)

The 6 linear equations can be written :

$$\begin{pmatrix} x_1^2 & y_1^2 & z_1^2 & y_1 z_1 & z_1 x_1 & x_1 y_1 & x_1 & y_1 & z_1 \\ \text{---} & & & & & & & & \\ \text{---} & & & & & & & & \\ \text{---} & & & & & & & & \\ \text{---} & & & & & & & & \\ x_6^2 & y_6^2 & z_6^2 & y_6 z_6 & z_6 x_6 & x_6 y_6 & x_6 & y_6 & z_6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ f \\ g \\ h \\ p \\ q \\ r \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

The 3 non linear equations can be solved simultaneously by the method of Newton. To find the initial value for the iterative calculation, the best method is to solve system (I) simply by adding a supplementary condition :

$\Sigma(a^2+b^2+ \dots +r^2)$ should be mini

The solution is obtained by inversing the system $AX = B$ to find a pseudo-inverse :

$$X = A^T (AA^T)^{-1} B$$

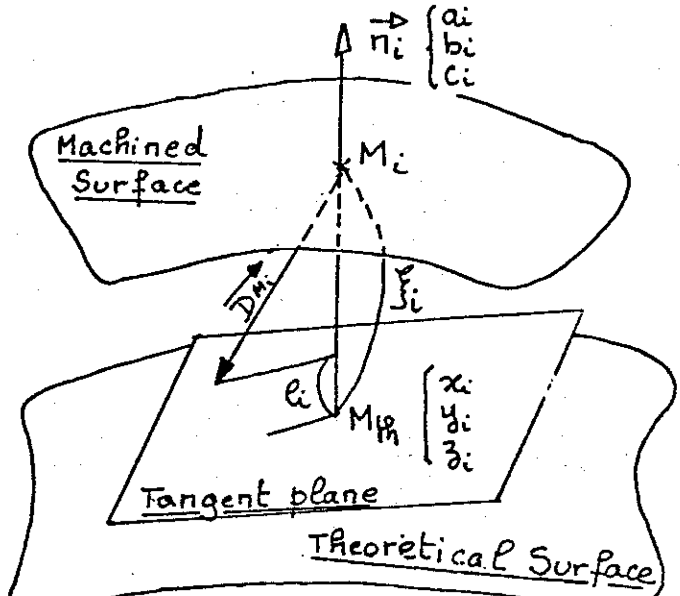
As matrix A has to be of maximum rank to calculate a pseudo-inverse, it means that the 6 measured points should be valid, i.e. that they are not in a particular configuration which is degenerated (e.g. 6 points in a plane are of rank 3). Here, matrix A has to be of rank 6 which is a useful criteria of validity of the measurements.

Special case : It is possible to identify a cylinder with only 5 points measured. This special case is developped page 6.

These methods are applicable to any surface identified by the general equation : $ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2px+2qy+2rz+d=0$ using the same general programme. However, the optimisation is carried out on the parameters of the equation and does not consider any physical aspect for describing the surface, i.e. the distance between measured points and the associated surface.

4. Method of Gauss applied to the displacement operator (torsor or screw) for small displacements

Deviations from the surface are measured for n points given by a table $(x_i, y_i, z_i, a_i, b_i, c_i)$ of the coordinates of the n points and the direction cosines of their normals. At point M_{th} (on the theoretical surface) with coordinates $x_i\ y_i\ z_i$ there is a normal to the theoretical surface.



M_i is a point of the machined surface obtained by intersection of normal n_i and the machined surface. An instrument of inspection will usually measure the algebric quantity :

$$\overline{M_i M_{th}} = \xi_i \quad \text{on normal } \vec{n_i}$$

The machined surface will never be exactly superposable by a displacement screw on the the theoretical surface, i.e. point M_i will not be coincident with point M_{th} (except for a number of points $n \leq 6$). But, it is possible to minimize the errors of superposition by imposing a displacement D such as to bring point M_i the nearest possible to the plane tangent to the theoretical surface in M_{th} . This can be expressed in algebric form as :

$$(1) \quad -\vec{D_{M_i}} \cdot \vec{n_i} + \xi_i = e_i$$

Let us define by $\mathcal{D}_A\{\vec{\Omega}, \vec{D_A}\}$ the screw of the small displacements which moves the machined surface from its actual position to a position so that it coincides in a significatibe way with the surface of definition. The displacement at point M_i is given by a relation similar to that of a field of moments taking A as the point of reference :

Which can also be rewritten :

$$\xi_i - \left[\vec{D}_A \cdot \vec{n}_i + (\vec{AM_i} \times \vec{n}_i) \cdot \vec{\Omega} \right) = e_i$$

This equation can be interpreted as the comoment of the displacement screw $\mathcal{D}_A$ and the plückerian coordinates of vector $\vec{n}_i$ $\quad \mathcal{P}_{A^i} \left\{ \begin{matrix} \vec{n}_i \\ \vec{AM_i} \times \vec{n}_i \end{matrix} \right.$

or in a condensed form : $\xi_i - (\mathcal{P}_{A^i} \cdot \mathcal{D}_A) = e_i$

This relation applies to every point of measurement. One derives a system of linear equations with 6 unknowns $(\alpha, \beta, \gamma, u, v, w)$, the 6 components of the displacement screw.

If the number of points defining the surface would only be 6 (general case) and reasonably chosen, one obtains a system of 6 equations, linear and independent with one solution corresponding to $e_1 = e_2 = \ldots = e_6 = 0$ or the superposition of 6 measured points to 6 points of definition. But, 6 points are not sufficient to define exactly the form of a surface of a part, and, practically, one uses a number of points superior to 6.

The system of equations is then solved by a method of Gauss by forming a function W , sum of e_i^2

$$W = \sum_{i=1}^{i=n} \left(\xi_i - \mathcal{P}_{Ai} \cdot \mathcal{D}_A \right)^2$$

This function has to be minimum for the best solution of the system of n equations. In general, a system of 6 equations is derived such as :

$$\frac{\partial W}{\partial \alpha} = 0 \quad \frac{\partial W}{\partial \beta} = 0 \quad \frac{\partial W}{\partial \gamma} = 0$$

$$\frac{\partial W}{\partial u} = 0 \quad \frac{\partial W}{\partial v} = 0 \quad \frac{\partial W}{\partial w} = 0$$

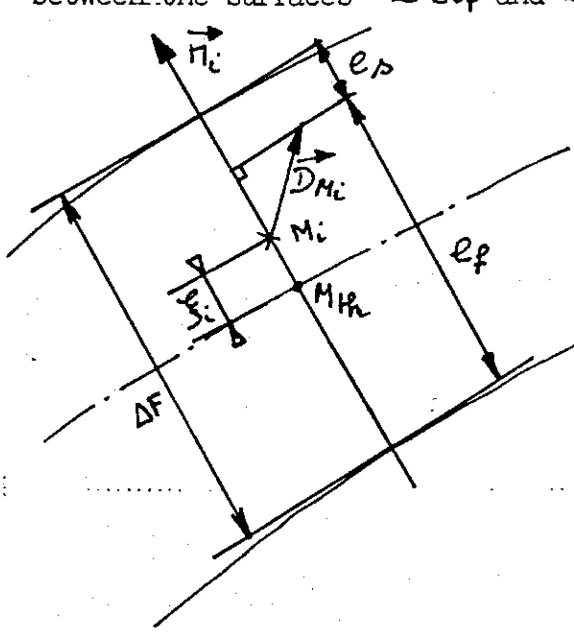
This system is linear and the solution is a screw $(\alpha, \beta, \gamma, u, v, w)$ which is a significative displacement and performs an optimal balancing of the part. By computation, the points are moved by the significative displacement screw and the new distance e_i to the tangent plane is determined, i.e. the form deviation between the machined surface and the definition surface.

This method is applicable to any form of surface which is defined by points and normals in the general case. The relevant equations are linear and the optimisation is carried out on a statistical distribution of form diviations Σ_i which is a physical reality and is not much influenced by erratic points. Furthermore, being linear, the system is resolved in an easy way.

5. Method of linear programming applied to the screw of small displacements and giving a minimal error of form

The machined surface defined by a set of points of measurements $(x_i\ y_i\ z_i)$. It is situated between two nominal surfaces S_{sup} (superior) and S_{inf} (inferior) parallel or concentric to the surface of definition and separated by the tolerance of form ΔF .

Let $\mathcal{T}_A \{ \vec{\Omega}, \vec{D}_A \}$ be the screw of small displacements moving the machined surface between the surfaces S_{sup} and S_{inf}



After displacement the distance e_s between point M_i and surface S_{sup} should be negative and the distance e_f between a point M_i and surface S_{inf} should be positive, or :

$$-\frac{\Delta F}{2} + \xi_i + \left(\mathcal{P}_{A_i} \cdot \mathcal{T}_A \right) \leq 0$$

and

$$\frac{\Delta F}{2} + \xi_i + \left(\mathcal{P}_{A_i} \cdot \tau_A \right) \geq 0$$

With the objective function ΔF of the tolerance of form being minimal. The resolution of the system of inequations will define a position of the machined surface which minimizes the deviation of form.

This method is in conformity with the international ISO 1101 standard and respects the physical nature of the error of form. It should be acceptable for the majority of metrologists and strandardisation institutions.

6. Conclusion

Although automatic threedimensional measuring machines are now used very commonly in industry, there is still a work of clarification to be pursued to evaluate exactly the efficiency of different methodologies of identification of surfaces.

In this paper, some of the most used methods of identification are described and interpreted in relation with their real physical signification. The superiority of the displacement methods is explained and the corresponding algorithms have been implemented on industrial machines(2).

It is the strong wish of the authors to pursue exchange of views on the identification methods among scientific metrodologists, as already undertaken by the Technical Committee "Quality assurance and metrology" of C.I.R.P. (International Institution for Production Engineering Research).

Special case :

It is possible to identify a cylinder with only 5 points measured if one accepts to ignore some special configurations. In the following derivation, one obtains X_o, Y_o, a, b and R by solving 5 non linear equations provided that no cylinder with an horizontal axis is considered.

Let M be a point on the axis of the cylinder with coordinates :

$$\vec{M} = \vec{M}_o + \lambda \vec{u}$$

$$\begin{cases} x = X_o + \lambda a \\ y = Y_o + \lambda b \\ z = 0 + \lambda \end{cases}$$

The equation of the circular cylinder with this axis would be obtained by eliminating λ :

$$\begin{cases} (x - X_o - \lambda a)^2 + (y - Y_o - \lambda b)^2 + (z - \lambda)^2 = R^2 \\ -a(x - X_o - \lambda a) - b(y - Y_o - \lambda b) - (z - \lambda) = 0 \\ \lambda(a^2 + b^2 + 1) = a(x - X_o) + b(y - Y_o) + z \end{cases}$$

$$\left[x - X_o - a\frac{[a(x-X_o) + b(y-Y_o) + z]}{a^2+b^2+1}\right]^2 +$$

$$\left[y - Y_o - b\frac{[a(x-X_o) + b(y-Y_o) + z]}{a^2+b^2+1}\right]^2 +$$

$$\left[z - \frac{a(x-X_o) + b(y-Y_o) + z}{a^2+b^2+1}\right]^2 = R^2$$

X_o, Y_o, a, b and R are then obtained by identificaiton with 5 measured points.

In the general case, when cylinder should pass through m points ($m > 6$) the method of the multipliers of Lagrange can be used to optimise with constraints.

Then, if Q_1 is the identification linear equation for a point M_i and F, G, H the 3 equations (non linear) which define a circular cylinder (equivalent to equations 1, 2 and 3 before), the following system has to be solved :

$$\begin{cases} Q_1 = 0 & F = 0 \\ Q_2 = 0 & G = 0 \\ \vdots & H = 0 \\ Q_m = 0 & \end{cases}$$

A function W is expressed :

$$W = \sum_{i=1}^{i=n} Q_i + \lambda F + \mu G + \nu H$$

by introducing the multipliers λ, μ, ν

and the following system is solved :

$$\begin{matrix} \dfrac{\partial W}{\partial a} = 0 & F = 0 \\ \vdots & G = 0 \\ \vdots & H = 0 \\ \dfrac{\partial W}{\partial R} = 0 & \end{matrix}$$

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